

# Moment estimates for convex measures

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## Abstract

Let  $p \geq 1$ ,  $\varepsilon > 0$ ,  $r \geq (1 + \varepsilon)p$ , and  $X$  be a  $(-1/r)$ -concave random vector in  $\mathbb{R}^n$  with Euclidean norm  $|X|$ . We prove that

$$(\mathbb{E}|X|^p)^{1/p} \leq c(C(\varepsilon)\mathbb{E}|X| + \sigma_p(X)),$$

where

$$\sigma_p(X) = \sup_{|z| \leq 1} (\mathbb{E}|\langle z, X \rangle|^p)^{1/p},$$

$C(\varepsilon)$  depends only on  $\varepsilon$  and  $c$  is a universal constant. Moreover, if in addition  $X$  is centered then

$$(\mathbb{E}|X|^{-p})^{-1/p} \geq c(\varepsilon)(\mathbb{E}|X| - C\sigma_p(X)).$$

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## 1 Introduction

Let  $X$  be a random vector with values in a finite dimensional Euclidean space  $E$  with Euclidean norm  $|\cdot|$  and scalar product  $\langle \cdot, \cdot \rangle$ . For any  $p > 0$ , we define the weak  $p$ -th moment of  $X$  by

$$\sigma_p(X) = \sup_{|z| \leq 1} (\mathbb{E}|\langle z, X \rangle|^p)^{1/p}.$$

Clearly  $(\mathbb{E}|X|^p)^{1/p} \geq \sigma_p(X)$  and by Hölder's inequality,  $(\mathbb{E}|X|^p)^{1/p} \geq \mathbb{E}|X|$ . In this paper we are interested in reversed inequalities of the form

$$(\mathbb{E}|X|^p)^{1/p} \leq C_1 \mathbb{E}|X| + C_2 \sigma_p(X) \tag{1}$$

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for  $p \geq 1$  and constants  $C_1$  and  $C_2$ .

This is known for some classes of distributions and the question has been studied in a more general setting (see [19] and references therein) and our objective in this paper is to describe new classes for which the relationship (1) is satisfied.

Let us recall some known results when (1) holds. It clearly holds for Gaussian vectors and it is not difficult to see that (1) is true for subgaussian vectors (see below for definitions) for every  $p \geq 1$ , with  $C_1$  and  $C_2$  depending only on the subgaussian parameter.

Another example of such a class is the class of so-called log-concave vectors. A probability measure  $\mu$  on  $\mathbb{R}^m$  is called log-concave if for all  $0 < \theta < 1$  and for all compact subsets  $A, B \subset \mathbb{R}^m$  with positive measure one has

$$\mu((1 - \theta)A + \theta B) \geq \mu(A)^{1-\theta} \mu(B)^\theta. \quad (2)$$

A random vector with a log-concave distribution is called log-concave. It is known that for every log-concave random vector  $X$  in a finite dimensional Euclidean space and any  $p > 0$ ,

$$(\mathbb{E}|X|^p)^{1/p} \leq C(\mathbb{E}|X| + \sigma_p(X)),$$

where  $C > 0$  is a universal constant. See Corollary 7 and references below.

In this paper we will consider the class of convex measures introduced by Borell. Let  $\kappa < 0$ . A probability measure  $\mu$  on  $\mathbb{R}^m$  is called  $\kappa$ -concave if for all  $0 < \theta < 1$  and for all compact subsets  $A, B \subset \mathbb{R}^m$  with positive measure one has

$$\mu((1 - \theta)A + \theta B) \geq ((1 - \theta)\mu(A)^\kappa + \theta\mu(B)^\kappa)^{1/\kappa}. \quad (3)$$

A random vector with a  $\kappa$ -concave distribution is called  $\kappa$ -concave. Note that a log-concave vector is also  $\kappa$ -concave for any  $\kappa < 0$ .

We show in Theorem 6 that for  $\kappa > -1$ , a  $\kappa$ -concave random vector satisfies (1) for all  $0 < (1 + \varepsilon)p < -1/\kappa$  with  $C_1$  and  $C_2$  depending only on  $\varepsilon$ .

In fact, in Definition 1 we will introduce a general assumption on the distribution, called  $H(p, \lambda)$ . The main result of the first part of the paper is Theorem 2 in which we show that this assumption is sufficient in order to have (1). In Theorem 5 we prove that convex measures satisfy this assumption.

One of the main applications of the relationship (1) consists in tail inequalities for  $\mathbb{P}(|X| \geq t \mathbb{E}|X|)$ . In Corollary 8 we show that for  $r > 2$  and for a  $(-1/r)$ -concave isotropic random vector  $X \in \mathbb{R}^n$  the above probability is bounded by  $\left(\frac{c \max\{1, r/\sqrt{n}\}}{t}\right)^{r/2}$ . From this bound we deduce that the empirical covariance matrix of a sample of size proportional to  $n$  is a good approximation of the covariance matrix of  $X$ , extending results of [1, 2] from log-concave measures to convex measures. This provides thus a new class of random vectors satisfying such approximation. See Corollary 10 and the remark following it.

The second part of the paper deals with negative moments. We are looking for relationship of the form

$$(\mathbb{E}|X|^{-p})^{-1/p} \geq C_1 \mathbb{E}|X| - C_2 \sigma_p(X) \quad (4)$$

for  $p > 0$  and constants  $C_1$  and  $C_2$ .

We show in Theorem 14 that for  $\kappa > -1$ , an  $n$ -dimensional  $\kappa$ -concave random vector satisfies (4) for all  $0 < (1 + \varepsilon)p < \min\{n/2, (-1/\kappa)\}$  with  $C_1$  and  $C_2$  depending only on  $\varepsilon$ . As an application we show a small ball probability estimate for  $\kappa$ -concave random vectors. In the log-concave setting it was proved in [27].

## 2 Preliminaries

The space  $\mathbb{R}^m$  is equipped with the scalar product  $\langle \cdot, \cdot \rangle$ , the Euclidean norm  $|\cdot|$ , the unit ball  $B_2^m$  and the volume measure  $\text{vol}(\cdot)$ . The canonical basis is denoted by  $e_1, e_2, \dots, e_m$ . A gauge or Minkowski functional  $\|\cdot\|$  on  $\mathbb{R}^m$  is a non-negative function on  $\mathbb{R}^m$  satisfying:  $\|\lambda x\| = \lambda\|x\|$  and  $\|x+y\| \leq \|x\| + \|y\|$  for every  $x, y \in \mathbb{R}^m$  and every real  $\lambda \geq 0$  and such that  $\|x\|=0$  if and only if  $x = 0$ . The dual gauge is defined for every  $x \in \mathbb{R}^m$  by  $\|x\|^* = \max\{\langle x, t \rangle : \|t\| \leq 1\}$ . A body is a compact subset of  $\mathbb{R}^m$  with a non-empty interior. Any convex body  $K \subset \mathbb{R}^m$  containing the origin in its interior defines the gauge by  $\|x\| = \inf\{\lambda \geq 0 : x \in \lambda K\}$ . It is called the Minkowski functional of  $K$ . If  $K \subset \mathbb{R}^m$  is a convex body containing the origin in its interior, the polar body  $K^\circ$  is defined by  $K^\circ = \{x \in \mathbb{R}^m : \langle x, t \rangle \leq 1 \text{ for all } t \in K\}$ . The diameter of  $K$  in the Euclidean metric is denoted by  $\text{diam}(K)$ .

For a linear subspace  $F \subset \mathbb{R}^n$  we denote the orthogonal projection on  $F$  by  $P_F$ . Note that  $P_F K^\circ := P_F(K^\circ) = (K \cap F)^\circ$ , when the polar is taken in  $F$ .

For a random vector  $X$  in  $\mathbb{R}^n$  with a density  $g$  and a subspace  $F \subset \mathbb{R}^n$ , we denote the density of  $P_F X$  by  $g_F$ .

A random vector  $X$  in  $\mathbb{R}^m$  will be called non-degenerated if it is not supported in a proper affine subspace of  $\mathbb{R}^m$ . It is called isotropic if it is centered and for all  $\theta \in \mathbb{R}^m$ ,  $\mathbb{E}|\langle X, \theta \rangle|^2 = |\theta|^2$ .

Given a non-negative bounded function  $g$  on  $\mathbb{R}^m$  we introduce the following associated set. For any  $\alpha \geq 1$ , let

$$K_\alpha(g) = \{t \in \mathbb{R}^m : g(t) \geq \alpha^{-m} \|g\|_\infty\}, \quad (5)$$

where  $\|g\|_\infty = \sup_{t \in \mathbb{R}^m} |g(t)|$ .

By  $g_i, g_{i,j}$  we denote independent standard Gaussian random variables, i.e. centered and of variance one. A standard Gaussian vector in  $\mathbb{R}^n$  is denoted by  $G$ , i.e.  $G = (g_1, g_2, \dots, g_n)$ . The standard Gaussian matrix is the matrix whose entries are i.i.d. standard Gaussian variables, i.e.  $\Gamma = \{g_{i,j}\}$ . By  $\gamma_p$  we denote the  $L_p$  norm of  $g_1$ . Note that  $\gamma_p/\sqrt{p} \rightarrow 1/\sqrt{e}$  as  $p \rightarrow \infty$ .

We denote by  $\mu_{n,k}$  the Haar probability measure on the Grassmannian  $G_{n,k}$  of  $k$ -dimensional subspaces of  $\mathbb{R}^n$ .

Recall that for a real number  $s$ ,  $\lceil s \rceil$  denotes the smallest integer which is not less than  $s$ .

By  $C, C_0, C_1, C_2, \dots, c, c_0, c_1, c_2$  we denote absolute positive constants, whose values can change from line to line.

For two functions  $f$  and  $g$  we write  $f \sim g$  if there are absolute positive constants  $c$  and  $C$  such that  $cf \leq g \leq Cf$ .

## 3 Convex probabilities

Let  $\kappa \leq 1/m$ . A Borel probability measure  $\mu$  on  $\mathbb{R}^m$  is called  $\kappa$ -concave if it satisfies (3). When  $\kappa = 0$ , this inequality should be read as (2) and it defines  $\mu$  as a log-concave probability.

In this paper we will be interested in the case  $\kappa \leq 0$ , which we consider from now on.

The class of  $\kappa$ -concave measures was introduced and studied by Borell. We refer to [9, 10] for a general study and to [8] for more recent development. A  $\kappa$ -concave probability is supported on some convex subset of an affine subspace where it has a density. When the support generates the whole space, a characterization of Borell ([9, 10]) states that the probability is absolutely continuous

with respect to the Lebesgue measure and has a density  $g$  which is log-concave when  $\kappa = 0$  and when  $\kappa < 0$ , is of the form

$$g = f^{-\beta} \quad \text{with} \quad \beta = m - \frac{1}{\kappa},$$

where  $f : \mathbb{R}^m \rightarrow (0, \infty]$  is a convex function. The class of  $m$ -dimensional  $\kappa$ -concave probabilities is increasing as  $\kappa$  is decreasing. In particular a log-concave probability is  $\kappa$ -concave for any  $\kappa < 0$ .

As we mentioned in the Introduction, a random vector with a  $\kappa$ -concave distribution is called  $\kappa$ -concave. Clearly, the linear image of a  $\kappa$ -concave probability is also  $\kappa$ -concave. Recall that any semi-norm of an  $m$ -dimensional vector with a  $\kappa$ -concave distribution has moments up to the order  $p < -1/\kappa$  (see [9] and Lemmas 21 and 22 below). Since we are interested in comparison of moments with the moment of order 1, we will always assume that  $-1 < \kappa \leq 0$ .

## 4 Strong and weak moments

In this section we consider a random vector  $X$  in a finite dimensional Euclidean space  $E$ .

**Definition 1.** Let  $p > 0$ ,  $m = \lceil p \rceil$ , and  $\lambda \geq 1$ . We say that a random vector  $X$  in  $E$  satisfies the assumption  $H(p, \lambda)$  if for every linear mapping  $A : E \rightarrow \mathbb{R}^m$  such that  $Y = AX$  is non-degenerate there exists a gauge  $\|\cdot\|$  on  $\mathbb{R}^m$  such that  $\mathbb{E}\|Y\| < \infty$  and

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq \lambda \mathbb{E}\|Y\|. \quad (6)$$

**Remark.** Let us give a first example of a random vector satisfying  $H(p, \lambda)$ . Let  $X$  be a random vector in an  $n$ -dimensional Euclidean space  $E$ , satisfying, for some  $\psi \geq 1$ ,

$$\forall z \in E \quad \forall 1 \leq p \leq n \quad (\mathbb{E}|\langle z, X \rangle|^p)^{1/p} \leq \psi \sqrt{p} \mathbb{E}|\langle z, X \rangle|. \quad (7)$$

Then  $X$  satisfies  $H(p, C\psi^2)$  for every  $p \geq 1$ . For example, the standard Gaussian and Rademacher vectors satisfy the above condition with  $\psi$  being a numerical constant. More generally, if  $X$  is subgaussian, then  $X$  satisfies (7).

To prove that (7) implies  $H(p, C\psi^2)$ , let  $p > 0$ ,  $m = \lceil p \rceil$  and let  $A : E \rightarrow \mathbb{R}^m$  be such that  $Y = AX$  is non-degenerate. We may assume that  $m \geq 2$ . Clearly, because of the linear invariance of the property (7), we may also assume that  $Y = AX$  is isotropic. Thus (7) yields,

$$\forall z \in \mathbb{R}^m \quad (\mathbb{E}|\langle z, Y \rangle|^p)^{1/p} \leq \psi \sqrt{p} \mathbb{E}|\langle z, Y \rangle| \leq \psi \sqrt{m} |z| \leq \sqrt{2} \psi^2 |z| \mathbb{E}|Y|, \quad (8)$$

where the last inequality follows from isotropicity of  $Y$  by applying (7) with  $p = 2$ ,  $z_i = A^\top e_i$ ,  $i \leq m$ , and the Cauchy-Schwarz inequality.

Now let us make the following general observation. Let  $p \geq 1$  and  $m = \lceil p \rceil$ . Let  $Y$  be a random vector in an  $m$ -dimensional normed space with norm  $\|\cdot\|$ . Since any  $m$ -dimensional norm can be estimated, up to a multiplicative constant, by the supremum over an exponential (in  $m$ ) number of norm one linear forms, we deduce that

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq C' \sup_{\|\varphi\|^* \leq 1} (\mathbb{E}|\varphi(Y)|^p)^{1/p}, \quad (9)$$

where  $C'$  is a universal constant (see [20] Proposition 3.20). Combining this with (8)

we conclude that

$$(\mathbb{E}|Y|^p)^{1/p} \leq C' \sup_{|z| \leq 1} (\mathbb{E}|\langle z, Y \rangle|^p)^{1/p} \leq C C' \psi \mathbb{E}|Y|,$$

which shows that  $X$  satisfies  $H(p, CC'\psi)$ .

The main result of this section states a relationship between weak and strong moments under the assumption  $H(p, \lambda)$ .

**Theorem 2.** *Let  $p > 0$  and  $\lambda \geq 1$ . If a random vector  $X$  in a finite dimensional Euclidean space satisfies  $H(p, \lambda)$ , then*

$$(\mathbb{E}|X|^p)^{1/p} \leq c(\lambda \mathbb{E}|X| + \sigma_p(X)),$$

where  $c$  is a universal constant.

The first step of the proof of Theorem 2 consists of showing that there exists some  $z$  such that  $(\mathbb{E}(\langle z, Y \rangle_+^p)^{1/p}$  is small, with comparison to  $\mathbb{E}|Y|$ . This is the purpose of the following lemma.

**Lemma 3.** *Let  $Y$  be a random vector in  $\mathbb{R}^m$ . Let  $\|\cdot\|_1$  and  $\|\cdot\|_2$  be two gauges on  $\mathbb{R}^m$  and  $\|\cdot\|_1^*$  and  $\|\cdot\|_2^*$  be their dual gauges. Then for all  $p > 0$ ,*

$$\min_{\|z\|_2^*=1} (\mathbb{E}(\langle z, Y \rangle_+^p)^{1/p} \leq \frac{(\mathbb{E}\|Y\|_1^p)^{1/p}}{\mathbb{E}\|Y\|_1} \mathbb{E}\|Y\|_2.$$

**Proof.** Let  $r$  be the largest real number such that  $r\|t\|_1 \leq \|t\|_2$  for all  $t \in \mathbb{R}^m$ . By duality  $r$  is the largest number such that  $r\|w\|_2^* \leq \|w\|_1^*$  for  $w \in \mathbb{R}^m$ . Pick  $z \in \mathbb{R}^m$  such that  $\|z\|_2^* = 1$  and  $\|z\|_1^* = r$ . Then  $\langle z, t \rangle \leq \|z\|_1^* \|t\|_1 \leq r\|t\|_1$  for all  $t \in \mathbb{R}^m$ . Therefore, for any  $p > 0$ ,  $(\mathbb{E}(\langle z, Y \rangle_+^p)^{1/p} \leq r(\mathbb{E}\|Y\|_1^p)^{1/p}$ . Thus the lemma follows from the inequality  $r\mathbb{E}\|Y\|_1 \leq \mathbb{E}\|Y\|_2$ .  $\square$

The second step of the proof of Theorem 2 is contained in the next lemma.

**Lemma 4.** *Let  $n, m \geq 1$  be integers. Let  $p \geq 1$ . Let  $X$  be an  $n$ -dimensional random vector and  $\Gamma$  be an  $n \times m$  standard Gaussian matrix. Then*

$$(\mathbb{E}|X|^p)^{1/p} \leq 2^{1/p} \gamma_p^{-1} \left( \mathbb{E} \min_{|t|=1} \|\Gamma t\|_{p,+} + (C\gamma_p + \sqrt{m}) \sigma_p(X) \right),$$

where  $\|z\|_{p,+} = (\mathbb{E}(\langle z, X \rangle_+^p)^{1/p}$  and  $C$  is a universal constant.

**Proof.** For every  $x, y \in \mathbb{R}^n$ ,  $|\|x\|_{p,+} - \|y\|_{p,+}| \leq |x - y| \sigma_p(X)$ . The classical Gaussian concentration inequality (see [12] or inequality (2.35) and Proposition 2.18 in [21]) gives that

$$\mathbb{P}(|\|G\|_{p,+} - \mathbb{E}\|G\|_{p,+}| \geq s) \leq 2 \exp(-s^2/2\sigma_p^2(X)),$$

and implies (cf. [22], Statement 3.1)

$$(\mathbb{E}\|G\|_{p,+}^p)^{1/p} \leq \mathbb{E}\|G\|_{p,+} + C\gamma_p \sigma_p(X), \quad (10)$$

where  $C$  is a universal constant. Since  $\langle G, X \rangle$  has the same distribution as  $|X|g_1$ ,

we have

$$\mathbb{E}(\langle G, X \rangle_+^p) = (1/2) \mathbb{E}|\langle G, X \rangle|^p \quad \text{and} \quad \mathbb{E}|\langle G, X \rangle|^p = \gamma_p^p \mathbb{E}|X|^p. \quad (11)$$

Therefore

$$(\mathbb{E}|X|^p)^{1/p} = 2^{1/p} \gamma_p^{-1} (\mathbb{E}\|G\|_{p,+}^p)^{1/p} \leq 2^{1/p} \gamma_p^{-1} (\mathbb{E}\|G\|_{p,+} + C\gamma_p\sigma_p(X)).$$

The Gordon minimax lower bound (see [15], Theorem 2.5) states that for any norm  $\|\cdot\|$

$$\mathbb{E}\|G\| \leq \mathbb{E} \min_{|t|=1} \|\Gamma t\| + \mathbb{E}|H| \max_{|z|=1} \|z\|,$$

where  $H$  is a standard Gaussian vector in  $\mathbb{R}^m$ . It is easy to check the proof and to show that this inequality remains true when  $\|\cdot\|$  is a gauge. This gives us that

$$\mathbb{E}\|G\|_{p,+} \leq \mathbb{E} \min_{|t|=1} \|\Gamma t\|_{p,+} + \mathbb{E}|H| \max_{|z|=1} \|z\|_{p,+} \leq \mathbb{E} \min_{|t|=1} \|\Gamma t\|_{p,+} + \sqrt{m} \max_{|z|=1} \|z\|_{p,+}$$

and it is enough to observe that  $\max_{|z|=1} \|z\|_{p,+} \leq \sigma_p(X)$ .  $\square$

**Proof of Theorem 2.** We may assume that  $p \geq 1$  since Theorem 2 is obviously true when  $0 < p \leq 1$ . Let  $m$  be the integer so that  $1 \leq p \leq m < p+1$ , thus  $m \leq 2p$ . We use the notation of Lemma 4. We first condition on  $\Gamma$ . We have

$$\|\Gamma z\|_{p,+} = (\mathbb{E}_X(\langle \Gamma z, X \rangle_+^p))^{1/p} = (\mathbb{E}_X(\langle z, \Gamma^* X \rangle_+^p))^{1/p}.$$

Let  $Y = \Gamma^* X \in \mathbb{R}^m$ . If  $Y$  is supported by a hyperplane then

$$\min_{|z|=1} (\mathbb{E}_X(\langle z, \Gamma^* X \rangle_+^p))^{1/p} = 0.$$

Otherwise by our assumption  $H(p, \lambda)$  there exists a gauge in  $\mathbb{R}^m$  such that

$$(\mathbb{E}\|Y\|^p)^{1/p} \leq \lambda \mathbb{E}\|Y\|.$$

From Lemma 3 we get

$$\min_{|z|=1} (\mathbb{E}_X(\langle z, \Gamma^* X \rangle_+^p))^{1/p} \leq \lambda \mathbb{E}_X|\Gamma^* X|.$$

We now take the expectation with respect to  $\Gamma$  and get

$$\mathbb{E} \min_{|t|=1} \|\Gamma t\|_{p,+} \leq \lambda \mathbb{E}|\Gamma^* X| = \lambda \mathbb{E}|H| \mathbb{E}|X| \leq \lambda \sqrt{m} \mathbb{E}|X|,$$

where  $H$  is a standard  $m$ -dimensional Gaussian vector. The proof is concluded using Lemma 4 and the fact that  $\gamma_p^{-1} \sqrt{p}$  is bounded. Indeed,

$$\begin{aligned} (\mathbb{E}|X|^p)^{1/p} &\leq 2^{1/p} \gamma_p^{-1} \lambda \sqrt{m} \mathbb{E}|X| + 2^{1/p} (C + \gamma_p^{-1} \sqrt{m}) \sigma_p(X) \\ &\leq c' (\lambda \mathbb{E}|X| + \sigma_p(X)). \end{aligned}$$

$\square$

## 5 Tail behavior of convex measures

**Theorem 5.** *Let  $n \geq 1$  and  $r > 1$ . Let  $X$  be a centered  $(-1/r)$ -concave random vector in a finite dimensional Euclidean space. Then for every  $0 < p < r$ ,  $X$  satisfies the assumption  $H(p, \lambda(p, r))$  with  $\lambda(p, r) = c \left(\frac{r}{r-1}\right)^3 \left(\frac{r}{r-p}\right)^4$ , where  $c$  is a universal constant.*

**Remark:** Note that the parameter  $\lambda(p, r)$  in Theorem 5 is bounded by a universal constant if the parameters  $p$  and  $r$  are not close, for instance if  $r \geq 2 \max\{1, p\}$ .

**Theorem 6.** *Let  $r > 1$  and let  $X$  be a  $(-1/r)$ -concave random vector in a finite dimensional Euclidean space. Then, for every  $0 < p < r$ ,*

$$(\mathbb{E}|X|^p)^{1/p} \leq c(C_2(p, r)\mathbb{E}|X| + \sigma_p(X)), \quad (12)$$

where  $C_2(p, r) = c \left( \frac{r}{r-1} \right)^3 \left( \frac{r}{r-p} \right)^4$  and  $c$  is a universal constant.

**Proof.** The proof may be reduced to the case of a centered random vector. Indeed, let  $X$  be a  $(-1/r)$ -concave random vector, then so is  $X - \mathbb{E}X$ . Since

$$(\mathbb{E}|X|^p)^{1/p} \leq (\mathbb{E}|X - \mathbb{E}X|^p)^{1/p} + |\mathbb{E}X| \leq (\mathbb{E}|X - \mathbb{E}X|^p)^{1/p} + \mathbb{E}|X|,$$

$\mathbb{E}|X - \mathbb{E}X| \leq 2\mathbb{E}|X|$  and  $\sigma_p(X - \mathbb{E}X) \leq 2\sigma_p(X)$ , we may assume that  $X$  is centered. The theorem now follows immediately by Theorems 2 and 5.  $\square$

Note that trivially a reverse inequality to (12) is valid, for  $p \geq 1$ :

$$2(\mathbb{E}|X|^p)^{1/p} \geq \mathbb{E}|X| + \sigma_p(X).$$

Therefore Theorem 6 states an equivalence

$$(\mathbb{E}|X|^p)^{1/p} \sim_{C_2(p, r)} \mathbb{E}|X| + \sigma_p(X).$$

Since a log-concave measure is  $\kappa$ -concave for any  $\kappa < 0$ , we obtain

**Corollary 7.** *For any log-concave random vector  $X$  in a finite dimensional Euclidean space and any  $p > 0$ ,*

$$(\mathbb{E}|X|^p)^{1/p} \leq C(\mathbb{E}|X| + \sigma_p(X)),$$

where  $C > 0$  is a universal constant.

As formulated here, Corollary 7 first appeared as Theorem 2 in [4] (see also [3]). A short proof of this result was given in [5]. It can be deduced directly from Paouris work in [26] (see [5]).

As it was mentioned above, if  $X \in E$  is  $(-1/r)$ -concave then so is  $\langle z, X \rangle$  for any  $z \in E$ . From Lemma 21, we have that for any  $1 \leq p < r$ ,

$$(\mathbb{E}|\langle z, X \rangle|^p)^{1/p} \leq C_1(p, r) \mathbb{E}|\langle z, X \rangle|, \quad (13)$$

where  $C_1(p, r)$  is defined in Lemma 21.

Assume that  $r > 2$ . Let  $n$  be the dimension of  $E$ . If moreover  $X$  is centered and has the identity as the covariance matrix – such a vector is called an *isotropic random vector* – then one has for any  $z \in S^{n-1}$  and any  $1 \leq p < r$ ,

$$(\mathbb{E}|\langle z, X \rangle|^p)^{1/p} \leq C_1(p, r) \mathbb{E}|\langle z, X \rangle| \leq C_1(p, r) (\mathbb{E}|\langle z, X \rangle|^2)^{1/2} = C_1(p, r). \quad (14)$$

Since in that case,  $\mathbb{E}|X| \leq (\mathbb{E}|X|^2)^{1/2} = \sqrt{n}$ , it follows from Theorem 6 that for any  $1 \leq p < r$ ,

$$(\mathbb{E}|X|^p)^{1/p} \leq c(C_2(p, r)\sqrt{n} + C_1(p, r)). \quad (15)$$

Together with Markov's inequality this give the following Corollary.

**Corollary 8.** *Let  $r > 2$  and let  $X \in \mathbb{R}^n$  be a  $(-1/r)$ -concave isotropic random vector. Then for every  $t > 0$ ,*

$$\mathbb{P}(|X| \geq t\sqrt{n}) \leq \left( \frac{c \max\{1, r/\sqrt{n}\}}{t} \right)^{r/2}. \quad (16)$$

*In particular, if  $r \geq 2\sqrt{n}$ , then for every  $6c \leq t \leq 3cr/\sqrt{n}$ ,*

$$\mathbb{P}(|X| \geq t\sqrt{n}) \leq \exp(-c_0 t\sqrt{n}), \quad (17)$$

*where  $c$  and  $c_0$  are universal positive constants.*

**Remark.** A log-concave measure is  $(-1/r)$ -concave for every  $r > 0$ , thus in such a case inequality (17) is valid for every  $t > c$ , which is a result from [26].

**Proof of Corollary 8.** The inequality (16) follows by Markov's inequality from inequality (15) with  $p = r/2$ , since  $C_2(r/2, r) \leq c$  and  $C_1(r/2, r) \leq cr$  for a universal positive  $c$ .

To prove the “In particular” part denote  $r' = t\sqrt{n}/(3c)$ . Note that  $r' \geq 2\sqrt{n}$  and that  $r' \leq r$ . Therefore  $X$  is  $(-1/r')$ -concave as well and we can apply (16) with  $r'$ , obtaining the bound for probability  $3^{-r'/2}$ , which implies the result.  $\square$

We now apply our results to the problem of the approximation of the covariance matrix by the empirical covariance matrix. Recall that for a random vector  $X$  the covariance matrix of  $X$  is given by  $\mathbb{E}XX^\top$ . It is equal to the identity operator  $I$  if  $X$  is isotropic. The empirical covariance matrix of a sample of size  $N$  is defined by  $\frac{1}{N} \sum_{i=1}^N X_i X_i^\top$ , where  $X_1, X_2, \dots, X_N$  are independent copies of  $X$ . The main question is how small  $N$  can be taken in order that these two matrices are close to each other in the operator norm (clearly, if  $X$  is non-degenerated then  $N \geq n$  due to the dimensional restrictions and, by the law of large numbers, the empirical covariance matrix tends to the covariance matrix as  $N$  grows to infinity). See [1, 2] for references on this question and for corresponding results in the case of log-concave measures. In particular, it was proved there that for  $N \geq n$  and log-concave  $n$ -dimensional vectors  $X_1, \dots, X_N$  one has

$$\left\| \frac{1}{N} \sum_{i=1}^N X_i X_i^\top - I \right\| \leq C \sqrt{\frac{n}{N}}$$

with probability at least  $1 - 2\exp(-c\sqrt{n})$ , where, as usual,  $I$  is the identity operator,  $\|\cdot\|$  is the operator norm  $\ell_2^n \rightarrow \ell_2^n$  and  $c, C$  are absolute positive constants.

In [29] (Theorem 1.1), the following condition was introduced: an isotropic random vector  $X \in \mathbb{R}^n$  is said to satisfy the *strong regularity assumption* if for some  $\eta, C > 0$  and every rank  $k \leq n$  orthogonal projection  $P$ , one has for every  $t \geq C$

$$\mathbb{P}(|PX| \geq t\sqrt{k}) \leq C t^{-2-2\eta} k^{-1-\eta}.$$

We show that an isotropic  $(-1/r)$ -concave random vector satisfies this assumption. For simplicity we will show this with  $\eta = 1$  (one can change  $\eta$  adjusting constants).

**Lemma 9.** *Let  $n \geq 1$ ,  $a > 0$  and  $r = \max\{4, 2a \log n\}$ . Let  $X \in \mathbb{R}^n$  be an isotropic  $(-1/r)$ -concave random vector. Then there exists an absolute constant*



$C$  such that for every rank  $k$  orthogonal projection  $P$  and every  $t \geq C_1(a)$ , one has

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq C_2(a) t^{-4} k^{-2},$$

where  $C_1(a) = C \exp(4/a)$  and  $C_2(a) = C \max\{(a \log a)^4, \exp(32/a)\}$ .

**Proof.** Let  $P$  be a projection of rank  $k$ . Let  $c$  be the constant from Corollary 8 (without loss of generality we assume  $c \geq 1$ ) and  $t > c$ . If  $r \leq \sqrt{k}$  then Corollary 8 implies that

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq \left(\frac{c}{t}\right)^{r/2} \leq \left(\frac{c}{t}\right)^{a \log n} = n^{-a \log(t/c)}.$$

If  $r > \sqrt{k}$  then  $X$  is also  $(-1/\sqrt{k})$ -concave, hence, assuming  $k > \max\{4, 64a^2 \log^2(4a)\}$  (so that  $\sqrt{k} \geq 2a \log k$ ) and applying Corollary 8 again we obtain

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq \left(\frac{c}{t}\right)^{\sqrt{k}/2} \leq \left(\frac{c}{t}\right)^{a \log k} = k^{-a \log(t/c)}.$$

Thus in both cases we have

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq k^{-a \log(t/c)}.$$

One can check that for  $t \geq c^2 \exp(4/a)$  and  $k \geq \exp(16/a)$  this implies

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq t^{-4} k^{-2},$$

which proves the desired result for  $k > C_a := \max\{64a^2 \log^2(4a), \exp(16/a)\}$  and  $t \geq c^2 \exp(4/a)$ .

Assume now that  $k \leq C_a$ . Then we apply Borell's Lemma – Lemma 21. We have that for every  $t \geq 3$

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq \left(1 + \frac{t}{9r}\right)^{-r}.$$

It is not difficult to see (e.g., by considering cases  $t \leq 9r$ ,  $9r < t \leq 18r$  and  $t > 18r$ ) that for  $C(a) := 54^4 C_a^2$ ,  $t \geq 3$  and  $r \geq 4$ , one has

$$\mathbb{P}\left(|PX| \geq t\sqrt{k}\right) \leq C(a) t^{-4} k^{-2}.$$

This completes the proof.  $\square$

Theorem 1.1 from [29] and the above lemma immediately imply the following corollary on the approximation of the covariance matrix by the empirical covariance matrix.

**Corollary 10.** *Let  $n \geq 1$ ,  $a > 0$  and  $r = \max\{4, 2a \log n\}$ . Let  $X_1, \dots, X_N$  be independent  $(-1/r)$ -concave isotropic random vectors in  $\mathbb{R}^n$ . Then for every  $\varepsilon \in (0, 1)$  and every  $N \geq C(\varepsilon, a)n$ , one has*

$$\mathbb{E} \left\| \frac{1}{N} \sum_{i=1}^N X_i X_i^\top - I \right\| \leq \varepsilon,$$

where  $C(\varepsilon, a)$  depends only on  $a$  and  $\varepsilon$ .

**Remark.** Let  $r = 2a \log(2n) > 8$ . Applying Corollary 8 for independent  $(-1/r)$ -concave isotropic random vectors  $X_1, X_2, \dots, X_N$  and using results of [24], it can be checked that with large probability

$$\left\| \frac{1}{N} \sum_{i=1}^N X_i X_i^\top - I \right\| \leq C(a) \sqrt{\frac{n}{N}}$$

where  $C(a)$  depends only on  $a$ . As we mentioned above, this extends the results of [1, 2] on the approximation of the covariance matrix from the log-concave setting to the class of convex measures.

Now we prove Theorem 5. We need the following lemma. Recall that  $K_\alpha$  was defined by (5).

**Lemma 11.** *Let  $m$  be an integer. Let  $r > 1$  and  $0 < p < r$ . Let  $Y \in \mathbb{R}^m$  be a centered random vector with density  $g = f^{-\beta}$  with  $\beta = m + r$  and  $f$  convex positive. Let  $F : \mathbb{R}^m \rightarrow \mathbb{R}^+$  be such that for every  $t \in \mathbb{R}^m$ ,  $F(2t) \leq 2^p F(t)$  and assume that  $\mathbb{E}F(Y)$  is finite. Then, there exists a positive universal constant  $c$  such that  $0 \in K_\alpha(g)$  and*

$$\mathbb{E}F(Y) \leq c(p, r) \mathbb{E} \left( F(Y) 1_{K_\alpha(g)}(Y) \right), \quad (18)$$

where  $c(p, r) = 1 + \frac{c}{r-p}$  and  $\alpha = \left( c \frac{(m+r)^2}{(r-p)(r-1)} \right)^{\frac{m+r}{m}}$ .

**Proof.** Let  $\alpha > 0$  be specified later. Let  $\gamma = \frac{\beta-m-1}{\beta-1}$ . From Lemma 20 we have  $\gamma f(0) \leq \min f = \|g\|_\infty^{-1/\beta}$  and by definition,  $\min f < \alpha^{-m/(r+m)} f(t)$  when  $t \notin K_\alpha(g)$ . Using the convexity of  $f$  and the last two inequalities we get

$$\forall t \notin K_\alpha(g) \quad g(t/2) \geq g(t) \left( \frac{1}{2} + \frac{1}{2} \gamma^{-1} \alpha^{-m/(r+m)} \right)^{-(r+m)}. \quad (19)$$

Let  $\delta = \delta(\alpha) := \left( 1 + \gamma^{-1} \alpha^{-m/(r+m)} \right)^{r+m}$ . The inequality (19) can be written

$$\forall t \notin K_\alpha(g) \quad g(t) \leq 2^{-r-m} \delta g(t/2).$$

Therefore

$$\mathbb{E}F(Y) 1_{K_\alpha(g)^c}(Y) \leq 2^{-r-m} \int_{K_\alpha(g)^c} F(t) g\left(\frac{t}{2}\right) \delta dt \leq 2^{-r} \int_{\mathbb{R}^m} F(2t) g(t) \delta dt$$

and from the assumption on  $F$ , we get

$$\mathbb{E}F(Y) 1_{K_\alpha(g)^c}(Y) \leq 2^{p-r} \delta \mathbb{E}F(Y).$$

We conclude that if  $2^{p-r} \delta < 1$  then

$$\mathbb{E}F(Y) \leq (1 - 2^{p-r} \delta)^{-1} \mathbb{E} \left( F(Y) 1_{K_\alpha(g)}(Y) \right).$$

Let

$$\alpha_0 = \left( \left( 2^{\frac{r-p}{2(r+m)}} - 1 \right) \gamma \right)^{-\frac{r+m}{m}},$$

so that  $\delta_0 := \delta(\alpha_0) = 2^{\frac{r-p}{2}}$ , then  $(1 - 2^{p-r} \delta_0)^{-1} = (1 - 2^{\frac{p-r}{2}})^{-1} \leq 1 + \frac{c}{r-p}$  and

$$\alpha_0 \leq \alpha = \left( c \frac{(r+m)^2}{(r-p)(r-1)} \right)^{\frac{r+m}{m}},$$

where  $c > 0$  is a universal constant. This concludes the proof of (18).

Clearly  $\gamma^{-1}\alpha^{-m/(r+m)} \leq \gamma^{-1}\alpha_0^{-m/(r+m)} < 1$  and recall that  $\gamma f(0) \leq \min f$ . We deduce that  $f(0) \leq \alpha^{\frac{m}{r+m}} \min f$  and thus  $0 \in K_\alpha(g)$ .  $\square$

**Remark.** An interesting setting for the previous lemma is when  $r$  is away from 1, for instance  $r \geq 2$ ,  $r$  and  $m$  are comparable, and  $p$  is proportional to  $r$ . In this case  $\gamma$  is bounded by a constant,  $c(p, r)$  explodes only when  $p \rightarrow r$ , and  $\alpha$  depends only on the ratio  $r/p$ .

**Proof of Theorem 5.** Let  $1 \leq p < r$  and  $m = \lceil p \rceil$ . Let  $A : E \rightarrow \mathbb{R}^m$  be a linear mapping and  $Y = AX$  be a centered non-degenerated  $(-1/r)$ -concave random vector. By Borell's result [9, 10], there exists a positive convex function  $f$  such that the distribution of  $Y$  has a density of the form  $g = f^{-(r+m)}$ .

We apply Lemma 11 and use the notation of that lemma. Because the class of  $(-1/r)$ -concave measures increases as the parameter  $r$  decreases, we may assume that  $r \leq 2p$  (note that  $\lambda(p, 2p) \sim \lambda(p, r)$  for  $r > 2p$ , so we do not lose control of the constant assuming that  $r \leq 2p$ ). Thus  $1 \leq p \leq m$  and  $r \leq 2m$ . We deduce that the parameter  $\alpha$  from Lemma 11 satisfies

$$\alpha \leq c \left( \frac{r}{r-1} \cdot \frac{r}{r-p} \right)^3,$$

where  $c$  is a numerical constant.

Now note that because  $g^{-1/(r+m)}$  is convex,  $K = K_\alpha(g)$  is a convex body and from Lemma 11, it contains 0. Let  $\|\cdot\|$  be its Minkowski functional.

We have

$$1 \geq \mathbb{P}(Y \in K) = \int_K g \geq \alpha^{-m} \|g\|_\infty \text{vol}(K),$$

so that

$$\mathbb{P}(\|Y\| \leq 1/(2\alpha)) = \int_{K/2\alpha} g \leq \|g\|_\infty (2\alpha)^{-m} \text{vol}(K) \leq 2^{-m} \leq 1/2,$$

and therefore

$$\mathbb{E}\|Y\| \geq \frac{1}{2\alpha} \mathbb{P}(\|Y\| > 1/(2\alpha)) \geq 1/(4\alpha).$$

Let  $F(t) = \|t\|^p$  for  $t \in \mathbb{R}^m$ . Thus  $F(2t) = 2^p F(t)$  and, since  $p < r$ ,  $\mathbb{E}F(Y)$  is finite. Hence  $F$  satisfies the assumption of Lemma 11. Therefore for  $c(p, r) = 1 + c/(r-p)$

$$\mathbb{E}\|Y\|^p \leq c(p, r) \mathbb{E}(\|Y\|^p 1_K(Y)) \leq c(p, r).$$

We conclude that

$$(\mathbb{E}\|Y\|^p)^{1/p} / \mathbb{E}\|Y\| \leq 4\alpha c(p, r)^{1/p} \leq c \left( \frac{r}{r-1} \right)^3 \left( \frac{r}{r-p} \right)^4$$

for some numerical constant  $c$ .  $\square$

Another application of Lemma 11, which will be used later, is the following lemma.

**Lemma 12.** Let  $1 \leq p < r$  and  $m = \lceil p \rceil$ . Let  $Y \in \mathbb{R}^m$  be a centered  $(-1/r)$ -concave random vector with density  $g$ . There exists a universal constant  $c$ , such that  $0 \in K_\alpha(g)$  and

$$(\mathbb{E}|\langle Y, t \rangle|^p)^{1/p} = \left( \int_{\mathbb{R}^m} |\langle x, t \rangle|^p g(x) dx \right)^{1/p} \leq C_3(p, r) \max_{x \in K_\alpha(g)} |\langle x, t \rangle|, \quad (20)$$

where  $\alpha = c \left( \frac{r^2}{(r-p)(r-1)} \right)^3$ ,  $C_3(p, r) = \left( 1 + \frac{c}{r-p} \right)^{1/p}$ , and  $c > 0$  is a universal constant.

**Proof.** Repeating the above proof with the function  $F(t) = |\langle x, t \rangle|^p$  we obtain that  $0 \in K_\alpha(g)$  and

$$\left( \int_{\mathbb{R}^m} |\langle x, t \rangle|^p g(x) dx \right)^{1/p} \leq \left( 1 + \frac{c}{r-p} \right)^{1/p} \left( \int_{K_\alpha(g)} |\langle x, t \rangle|^p g(x) dx \right)^{1/p}.$$

Clearly

$$\left( \int_{K_\alpha(g)} |\langle x, t \rangle|^p g(x) dx \right)^{1/p} \leq \max_{x \in K_\alpha(g)} |\langle x, t \rangle| \left( \int_{\mathbb{R}^m} g(x) dx \right)^{1/p} = \max_{x \in K_\alpha(g)} |\langle x, t \rangle|,$$

which implies the result.  $\square$

## 6 Small ball probability estimates

The following result was proved in [27].

**Theorem 13.** *Let  $X$  be a centered log-concave random vector in a finite dimensional Euclidean space. For every  $\varepsilon \in (0, c')$  one has*

$$\mathbb{P} \left( |X| \leq \varepsilon (\mathbb{E}|X|^2)^{1/2} \right) \leq \varepsilon^{c(\mathbb{E}|X|^2)^{1/2}/\sigma_2(X)},$$

where  $c, c' > 0$  are universal positive constants.

In this section we generalize this result to the setting of convex distributions. We first establish a lower bound for the negative moment of the Euclidean norm of a convex random vector.

**Theorem 14.** *Let  $r > 1$  and let  $X$  be a centered  $n$ -dimensional  $(-1/r)$ -concave random vector. Assume  $1 \leq p < \min\{r, n/2\}$ . Then*

$$(\mathbb{E}|X|^{-p})^{-1/p} \geq C_4(p, r) (\mathbb{E}|X| - C\sigma_p(X)),$$

where

$$C_4(p, r) = c \left( \frac{r^2}{(r-p)(r-1)} \right)^{-3} \left( 1 + \frac{c}{r-p} \right)^{-1/p}$$

and  $c, C$  are absolute positive constants. Moreover, if  $0 < p < 1$  then

$$(\mathbb{E}|X|^{-p})^{-1/p} \geq c_0 (1-p) \frac{r-1}{r^2} \mathbb{E}|X|,$$

where  $c_0$  is an absolute positive constant.

From Markov's inequality we deduce a small ball probability estimates for convex measures.

**Theorem 15.** *Let  $n \geq 1$  and  $r > 1$ . Let  $X$  be a centered  $n$ -dimensional  $(-1/r)$ -concave random vector. Assume  $1 \leq p < \min\{r, n/2\}$ . Then, for every  $\varepsilon \in (0, 1)$ ,*

$$\mathbb{P}(|X| \leq \varepsilon \mathbb{E}|X|) \leq (2C_4^{-1}(p, r)\varepsilon)^p,$$

whenever  $\mathbb{E}|X| \geq 2C\sigma_p(X)$ , where  $c, C$  and  $C_4(p, r)$  are the constants from Theorem 14.

**Remark.** Theorem 15 implies Theorem 13 proved in [27]. Indeed, let  $p \geq 1$ ,  $r \geq \max\{3, 2p\}$  and  $A := (\mathbb{E}|X|^2)^{1/2}/\sigma_2(X)$  (note that  $A \leq \sqrt{n}$ ). By Lemma 21,

$$\sigma_p(X) \leq C_1(p, r)\sigma_1(X) \leq c_0 p \sigma_2(X) \quad \text{and} \quad (\mathbb{E}|X|^2)^{1/2} \leq c_1 \mathbb{E}|X|.$$

Thus  $\mathbb{E}|X|/\sigma_p(X) \geq c_2 A/p$ . If  $c_2 A/(2C) \geq 1$ , we chose  $p = c_2 A/(2C)$  and apply Theorem 15. Since

$$\mathbb{E}|X|/\sigma_p(X) \geq c_2 A/p \geq 2C,$$

Theorem 13 follows. Now assume that  $A \leq 2C/c_2$ . Then Theorem 13 follows from Lemma 23 and Lemma 22 (with  $q = 2$ ), which for a log-concave random vector  $X$  states that  $(\mathbb{E}|X|^2)^{1/2} \leq c \text{Med}(|X|)$ , where  $c$  is a numerical constant and  $\text{Med}(|X|)$  is a median of  $|X|$ .

We need the following result from [18] (Theorem 1.3 there).

**Theorem 16.** *Let  $n \geq 1$  be an integer,  $\|\cdot\|$  be a norm in  $\mathbb{R}^n$  and  $K$  be its unit ball. Assume that  $0 < p \leq c_0 (\mathbb{E}\|G\|/\sigma)^2$  and  $m = \lceil p \rceil$ . Then*

$$\frac{c \mathbb{E}\|G\|}{\sqrt{n}} \leq \left( \int_{G_{n,m}} (\text{diam}(K \cap F))^m d\mu(F) \right)^{-1/m} \leq \frac{\mathbb{E}\|G\|}{c\sqrt{n}},$$

where  $\mu = \mu_{n,m}$  and  $c$  is an absolute positive constant.

The proof of Theorem 14 is based on the following two lemmas.

**Lemma 17.** *Let  $m \leq n$ ,  $\alpha > 0$  and  $X$  be a random vector in  $\mathbb{R}^n$  with density  $g$ . Then,*

$$(\mathbb{E}|X|^{-m})^{-1/m} \geq \frac{1}{\sqrt{2\pi\alpha}} (\mathbb{E}|G|^{-m})^{-1/m} \left( \int_{G_{n,m}} (\text{vol}(K_\alpha(g_F)))^{-1} d\mu(F) \right)^{-1/m}.$$

**Proof.** Integrating in polar coordinates (see [27], Proposition 4.6), we obtain the following key formula

$$(\mathbb{E}|X|^{-m})^{-1/m} = (2\pi)^{-1/2} (\mathbb{E}|G|^{-m})^{-1/m} \left( \int_{G_{n,m}} g_F(0) d\mu(F) \right)^{-1/m}.$$

Note that

$$1 = \int_F g_F(x) dx \geq \int_{K_\alpha(g_F)} g_F(x) dx \geq \alpha^{-m} \|g_F\|_\infty \text{vol}(K_\alpha(g_F)).$$

This implies the result, since  $g_F(0) \leq \|g_F\|_\infty$ .  $\square$

Below we will use the following notation. For a random vector  $X$  in  $\mathbb{R}^n$ ,  $p > 0$ , and  $t \in \mathbb{R}^n$  we denote

$$\|t\|_p = (\mathbb{E}|\langle X, t \rangle|^p)^{1/p}$$

(note that it is the dual gauge of the so-called centroid body, which is rather an  $L_p$ -norm than the  $\ell_p$ -norm).

**Lemma 18.** *Let  $1 \leq p < r$  and  $m = \lceil p \rceil$ . Let  $X$  be a centered  $(-1/r)$ -concave random vector in  $\mathbb{R}^n$  with density  $g$ . Let  $K$  denote the unit ball of  $\|\cdot\|_p$ . Then for every  $m$ -dimensional subspace  $F \subset \mathbb{R}^n$  one has*

$$(\text{vol}(P_F K^\circ))^{1/m} \leq 4C_3(p, r)(\text{vol}(K_\alpha(g_F)))^{1/m},$$

where  $\alpha = c \left( \frac{r^2}{(r-p)(r-1)} \right)^3$ ,  $C_3(p, r) = \left( 1 + \frac{c}{r-p} \right)^{1/p}$ , and  $c > 0$  is a universal constant.

**Proof.** Applying Lemma 12 to  $Y = P_F X$ , we obtain that for every  $t \in F$

$$\|t\|_p \leq C_3(p, r) \max_{x \in K_\alpha(g_F)} |\langle x, t \rangle|$$

with  $\alpha$  and  $C_3(p, r)$  given in Lemma 12. Since for  $t \in F$ ,  $\|t\|_p = \max \langle x, t \rangle$ , where the supremum is taken over  $x \in (K \cap F)^\circ = P_F K^\circ$ , this is equivalent to

$$P_F K^\circ \subset C_3(p, r) \text{conv}(K_\alpha(g_F) \cup -K_\alpha(g_F)).$$

Lemma 12 also claims that  $0 \in K_\alpha(g_F)$ , thus

$$\text{conv}(K_\alpha(g_F) \cup -K_\alpha(g_F)) \subset K_\alpha(g_F) - K_\alpha(g_F).$$

By Rogers-Sheppard inequality [28] we observe

$$(\text{vol}(P_F K^\circ))^{1/m} \leq \binom{2m}{m}^{1/m} C_3(p, r)(\text{vol}(K_\alpha(g_F)))^{1/m}.$$

This implies the result.  $\square$

**Proof of Theorem 14.** Recall that  $c_1, c_2, \dots$  denote absolute positive constants. Recall also that for a random vector  $X$  in  $\mathbb{R}^n$ ,  $p > 0$ , and  $t \in \mathbb{R}^n$

$$\|t\|_p = (\mathbb{E}|\langle X, t \rangle|^p)^{1/p}.$$

Given a norm  $\|\cdot\|$  on  $\mathbb{R}^n$  we define  $\sigma$  by  $\sigma = \sigma(\|\cdot\|) = \max_{|t|=1} \|t\|$ . In particular,

$$\sigma(\|\cdot\|_p) = \sigma_p(X).$$

Finally, let  $K$  denote the unit ball of  $\|\cdot\|_p$ .

We assume that  $X$  is non-degenerate in  $\mathbb{R}^n$  and let  $m = \lceil p \rceil$ . Without loss of generality we assume that

$$\mathbb{E}|X| \geq C\sigma_p(X),$$

where  $C$  is a large enough absolute constant.

As in (10), since  $p \leq m \leq 2p$ , we have

$$\begin{aligned} \mathbb{E}|X| &\leq (\mathbb{E}|X|^p)^{1/p} = \gamma_p^{-1}(\mathbb{E}\|G\|_p^p)^{1/p} \leq \gamma_p^{-1}(\mathbb{E}\|G\|_p + c_1\gamma_p\sigma_p(X)) \\ &\leq c_2(\mathbb{E}\|G\|_p/\sqrt{m} + \sigma_p(X)). \end{aligned}$$

Hence

$$\mathbb{E}\|G\|_p \geq \sqrt{p}c_2^{-1}(\mathbb{E}|X| - c_2\sigma_p(X)) \geq \sqrt{p}(c_2)^{-1}(C - c_2)\sigma_p(X). \quad (21)$$

This implies that for sufficiently large  $C$  we have  $m \leq 2p \leq c_0(\mathbb{E}\|G\|_p/\sigma_p(X))^2$ , where  $c_0$  is the constant from Theorem 16.

Note that  $(\mathbb{E}|G|^{-p})^{-1/p} \geq (\mathbb{E}|G|^{-m})^{-1/m} \geq c_3\sqrt{n}$  (the second inequality is well known for  $m \leq n/2$  and can be directly computed). Combining Lemmas 17 and 18, we obtain

$$(\mathbb{E}|X|^{-m})^{-1/m} \geq \frac{c_4\sqrt{n}}{\alpha C_3(p, r)} \left( \int_{G_{n,m}} (\text{vol}(P_F K^\circ))^{-1} d\mu(F) \right)^{-1/m},$$

with  $\alpha$  and  $C_3(p, r)$  as in Lemma 18.

Now note that  $P_F K^\circ = (K \cap F)^\circ \supset (\text{diam}(K \cap F))^{-1} B_2^n \cap F$ . Therefore  $1/\text{vol}(P_F K^\circ) \leq (c_5\sqrt{m} \text{diam}(K \cap F))^m$ . Applying Theorem 16, we obtain

$$(\mathbb{E}|X|^{-m})^{-1/m} \geq \frac{c_6}{\alpha C_3(p, r)\sqrt{m}} \mathbb{E}\|G\|_p.$$

Applying the first inequality from (21), we obtain the desired result.

The ‘‘Moreover’’ part is an immediate corollary of Lemmas 22 (with  $q = 1$ ) and 23.  $\square$

**Conjecture.** We conjecture that for convex distributions a similar thin shell property holds as for log-concave distribution: if  $X$  is an isotropic  $(-1/r)$ -concave random vector in  $\mathbb{R}^n$  with  $r > 2$ , then

$$\forall t \in (0, 1) \quad \mathbb{P}(|X| - \mathbb{E}|X| \geq t\sqrt{n}) \rightarrow 0.$$

as  $n$  tends to  $\infty$ . See [17] for recent work in the log-concave setting.

## 7 Appendix

There is a vast literature on inequalities of integrals related to concave functions. Some of the following lemmas may be known but we did not find any reference. Their proof use classical methods for demonstrating integral inequalities involving concave functions (see [11] and [25]). The first lemma is a mirror image for negative moments of a result from [23] valid for positive moments.

**Lemma 19.** *Let  $s, m, \beta \in \mathbb{R}$  such that  $\beta > m + 1 > 0$  and  $s > 0$ . Let  $\varphi$  be a non-negative concave function on  $[s, +\infty)$ . Then*

$$G(\beta) = \frac{\int_s^\infty \varphi^m(x) x^{-\beta} dx}{s^{m-\beta+1} B(m+1, \beta-m-1)}$$

*is an increasing function of  $\beta$  on  $(m+1, \infty)$ . Here  $B(u, v) = \int_0^1 (1-t)^{u-1} t^{v-1} dt$  denotes the Beta function.*

**Proof.** Let  $\beta > m + 1$ . Consider the function

$$H(t) = \int_s^t \varphi^m(x) x^{-\beta} dx - \int_s^t a^m (x-s)^m x^{-\beta} dx$$

for  $t \geq s$ , where  $a$  is chosen so that  $H(\infty) = 0$ . Note that  $H'$ , the derivative of  $H$ , has the same sign as  $(\varphi(x)/(x-s))^m - a^m$ . Since  $\varphi(x)/(x-s)$  is decreasing on  $(s, +\infty)$ , we deduce that  $H$  is first increasing and then decreasing. Since  $H(s) = H(\infty) = 0$  we conclude that  $H$  is non-negative. This means that for every  $t \geq s$ ,

$$\int_s^t \varphi^m(x) x^{-\beta} dx \geq \int_s^t a^m (x-s)^m x^{-\beta} dx. \quad (22)$$

Now, note that for any  $\beta' > \beta$  and any non-negative function  $F$ , we have by Fubini's theorem,

$$\int_s^\infty F(x)x^{-\beta'} dx = \int_s^\infty (\beta' - \beta)t^{-\beta'+\beta-1} \left( \int_s^t F(x)x^{-\beta} dx \right) dt.$$

Using (22) and applying this relation to  $F = \varphi^m$  and then to  $F(x) = a^m(x-s)^m$ , we get that

$$\int_s^\infty \varphi^m(x)x^{-\beta'} dx \geq a^m \int_s^\infty (x-s)^m x^{-\beta'} dx.$$

From the definition of  $a$ , we conclude that

$$\int_s^\infty \varphi^m(x)x^{-\beta} dx / \int_s^\infty (x-s)^m x^{-\beta} dx$$

is an increasing function of  $\beta$  on  $(m+1, \infty)$ . The conclusion follows from the computation of  $\int_s^\infty (x-s)^m x^{-\beta} dx = s^{m-\beta+1} B(m+1, \beta-m-1)$ .  $\square$

**Lemma 20.** *Let  $m \geq 1$  be an integer. Let  $g$  be the density of a probability on  $\mathbb{R}^m$  of the form  $g = f^{-\beta}$  with  $f$  positive convex on  $\mathbb{R}^m$  and  $\beta > m+1$ . If  $\int xg(x) dx = 0$ , then*

$$g(0) \geq \left( \frac{\beta-m-1}{\beta-1} \right)^\beta \|g\|_\infty.$$

**Proof.** Since  $f$  is convex it follows from Jensen's inequality that

$$f(0) = f \left( \int xg(x) dx \right) \leq \int f^{-\beta+1}(x) dx = \int_s^\infty (\beta-1)h(s)s^{-\beta} ds,$$

where  $s = \min f = \|g\|_\infty^{-1/\beta}$  and  $h(t) = \text{vol}\{f \leq t\}$  denotes the Lebesgue measure of  $\{f \leq t\}$ . From the convexity of  $f$  and from the Brunn-Minkowski inequality,  $\varphi = h^{1/m}$  is concave. Thus, using the notation of Lemma 19,

$$f(0) \leq (\beta-1)s^{m-\beta+1} B(m+1, \beta-m-1) G(\beta).$$

Now observe that  $\int f^{-\beta} = \int_s^\infty \beta \varphi^m(x)x^{-\beta-1} dx = 1$  and therefore, by Lemma 19,

$$G(\beta) \leq G(\beta+1) = (\beta s^{m-\beta} B(m+1, \beta-m))^{-1}.$$

The conclusion follows from combining the last two inequalities.  $\square$

**Remark.** When  $\beta \rightarrow \infty$ , which corresponds to a log-concave density, we recover the inequality from [13] saying that  $g(0) \geq e^{-m} \|g\|_\infty$ .

The next lemma is a well known result of Borell ([9]) stated in a way that fits our needs and stresses the dependence on the parameter of concavity.

**Lemma 21.** *Let  $r > 1$  and  $X$  be a  $(-1/r)$ -concave random vector in  $\mathbb{R}^m$ . Then for any semi-norm  $\|\cdot\|$  and any  $t \geq 1$ , one has*

$$\mathbb{P}(\|X\| \geq 3t\mathbb{E}\|X\|) \leq \left( 1 + \frac{t}{3r} \right)^{-r}.$$

As a consequence, for every  $1 \leq p < r$ ,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C_1(p, r) \mathbb{E}\|X\|,$$

where  $C_1(p, r) = cp$  for  $r > p+1$ ,  $C_1(p, r) = \frac{cr}{(r-p)^{1/p}}$  otherwise and  $c$  is a universal constant.



**Proof.** Denote  $\theta := \mathbb{P}(\|X\| \leq 3\mathbb{E}\|X\|)$ . Assume that  $\theta < 1$  (otherwise we are done). From Markov's inequality,

$$\theta = 1 - \mathbb{P}(\|X\| > 3\mathbb{E}\|X\|) \geq 2/3.$$

The subset  $B = \{x \in \mathbb{R}^m : \|x\| \leq 3\mathbb{E}\|X\|\}$  is symmetric and convex. From Lemma 3.1 in [9], for every  $t \geq 1$ , one has

$$\mathbb{P}(\|X\| \geq 3t\mathbb{E}\|X\|) \leq \left( \frac{t+1}{2} \left( (1-\theta)^{-1/r} - \theta^{-1/r} \right) + \theta^{-1/r} \right)^{-r}.$$

Thus,

$$\mathbb{P}(\|X\| \geq 3t\mathbb{E}\|X\|) \leq \theta \left( 1 + \frac{1}{2r} \log \frac{\theta}{1-\theta} + \frac{t}{2r} \log \frac{\theta}{1-\theta} \right)^{-r}.$$

We deduce that for every  $t \geq 1$ ,

$$\mathbb{P}(\|X\| \geq 3t\mathbb{E}\|X\|) \leq \theta \left( 1 + \frac{t}{2r} \log \frac{\theta}{1-\theta} \right)^{-r} \leq \left( 1 + \frac{t}{3r} \right)^{-r}.$$

Integrating, we get

$$\begin{aligned} \mathbb{E}\|X\|^p / (3\mathbb{E}\|X\|)^p &= \int_0^\infty pt^{p-1} \mathbb{P}(\|X\| \geq 3t\mathbb{E}\|X\|) dt \\ &\leq 1 + \int_1^\infty pt^{p-1} \left( 1 + \frac{t}{3r} \right)^{-r} dt \\ &\leq 1 + (3r)^p pB(p, r-p) \\ &= 1 + (3r)^p \Gamma(p+1)\Gamma(r-p)/\Gamma(r). \end{aligned}$$

Now, if  $r > p+1$  then, by Stirling's formula,

$$(\Gamma(p+1)\Gamma(r-p)/\Gamma(r))^{1/p} \sim \frac{p}{r},$$

and if  $r \leq p+1$  then

$$(\Gamma(p+1)\Gamma(r-p)/\Gamma(r))^{1/p} \sim (\Gamma(r-p))^{1/p} \sim \left( \frac{1}{r-p} \right)^{1/p}.$$

This completes the proof.  $\square$

The following, a stronger variant of Borell's lemma, allows to compare the expectation of a random variable  $\|X\|$  and a median  $\text{Med}(\|X\|)$ . It was proved in [7] (Theorem 1.1, see also the discussion following Theorem 5.2 for the behavior of the corresponding constant). It was also implicitly proved in [16] (see inequality (4) in [14]). The second part of the lemma follows by integration.

**Lemma 22.** *Let  $r > 1$  and  $X$  be a  $(-1/r)$ -concave random vector in  $\mathbb{R}^m$ . Then for any semi-norm  $\|\cdot\|$  and any  $t \geq 1$ , one has*

$$\mathbb{P}(\|X\| \geq t\text{Med}(\|X\|)) \leq (C_0 r)^r t^{-r},$$

where  $C_0$  is an absolute positive constant. As a consequence, for every  $r > q \geq 1$  one has

$$(\mathbb{E}\|X\|^q)^{1/q} \leq Cr \left( \frac{r}{r-q} \right)^{1/q} \text{Med}(\|X\|),$$

where  $C$  is an absolute positive constant.

The following lemma is Corollary 9 from [14] (as before, the second part follows by integration).

**Lemma 23.** *Let  $r > 1$  and  $X$  be a  $(-1/r)$ -concave random vector in  $\mathbb{R}^m$ . Then for any semi-norm  $\|\cdot\|$  and any  $\varepsilon \in (0, 1)$ , one has*

$$\mathbb{P}(\|X\| \leq \varepsilon \text{Med}(\|X\|)) \leq C_0 \varepsilon,$$

where  $C_0$  is an absolute positive constant. As a consequence, for every  $p \in (0, 1)$ ,

$$(\mathbb{E}\|X\|^{-p})^{-1/p} \geq c(1-p)\text{Med}(\|X\|),$$

where  $c$  is an absolute positive constant.

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